JOURNAL OF AIRCRAFT
Vol. 33, No. 5, September–October 1996

Engineering Notes

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Why an Engine Slows Down at Onset of Climb

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Introduction

PILOTS of fixed-pitch propeller-driven aircraft, as well as engineers (e.g., Weick[1]) dealing with this type of aircraft, know that as one transitions from level cruise to climb, leaving the throttle (manifold pressure) fixed, engine speed n drops off as airspeed V decreases. A question is: why? And, for the quantitatively inclined: how much? This Note employs classical reductio ad absurdum to answer the first question and elementary Jacobian manipulations to answer the second. The analytic answer to the second question, a formula for $\mathrm{d}n/\mathrm{d}V$, is apparently new and should be useful in aircraft performance studies. Both results, because of their relative simplicity, may be of pedagogic interest.

Assumptions

Assume propeller and engine behavior are governed by an ordinary steady-state performance model in which the following occurs:

1) Air density ρ (or relative air density σ) is constant. Assume transition from cruise to climb occurs over a time short enough that gross altitude changes are negligible.

2) Propeller behavior is modeled by the usual dimensionless thrust and power coefficient functions $C_T(J)$ and $C_P(J)$. The variable J is the propeller advance ratio V/nd and d is the propeller diameter. Moreover, the slope of $C_P(J)$ in the working range (not including early portions of the takeoff run, or diving under power), is negative:

$$\frac{\mathrm{d}C_P(J)}{\mathrm{d}J} < 0 \tag{1}$$

3) Constant throttle is tantamount to constant engine torque $Q = P/2\pi n$, where P is brake or shaft power. In fact, torque increases a bit when the engine is loaded down (n diminishing), but not by much. For a typical 160-hp general aviation engine, Q increases only 4% over a 700 rpm decrease in engine speed.

When V Decreases, So Does n

A direct qualitative argument, made with the hope of uncovering some indubitable fact from which one can successfully argue backwards, can be seen to fail. Assume that as airspeed V decreases, so does engine speed n. From this, using the definition of advance ratio J, nothing can be told about whether J increases or decreases or stays constant.

So an indirect argument is made. As climb starts (and V decreases), either n decreases or it doesn't. Assume it doesn't. Then n either increases or stays constant. In either of those cases, J decreases. Then (by assumption 2) power coefficient C_P increases. But by the definition of C_P and assumption 3, we have

$$C_P \equiv \frac{P}{\rho n^3 d^5} = \frac{2\pi Q}{\rho n^2 d^5} = \frac{K}{n^2} \tag{2}$$

which defines constant K and proves that n must decrease. So if n either increases or stays constant, then n decreases. Therefore, when V decreases, n in fact decreases, as observed in flight. The next section offers a more constructive argument.

Quantitative Solution

This is a classical advanced calculus (e.g., Kaplan[2]), problem of finding $\mathrm{d}n/\mathrm{d}V$ when there are four variables: one independent (V) and three dependent (n, C_P, and J). These four variables are related through the following three equations:

1) From Eq. (2)

$$F(n, C_P) \equiv C_P - (K/n^2) = 0 \tag{3}$$

2) So as not to confuse any current (cruise or climb) value of C_P with its functional form, call the latter $f(J)$. Then

$$G(C_P, J) \equiv C_P - f(J) = 0 \tag{4}$$

3) Finally, the definition of advance ratio gives

$$H(n, V, J) \equiv J - (V/nd) = 0 \tag{5}$$

With the ground work laid, it's now a matter for direct calculation. Using a common notation for Jacobian determinants

$$\frac{\mathrm{d}n}{\mathrm{d}V} = -\left[\frac{\partial(F, G, H)}{\partial(V, C_P, J)}\right] \Bigg/ \left[\frac{\partial(F, G, H)}{\partial(n, C_P, J)}\right] \tag{6}$$

Working out the various partial derivatives gives

$$\frac{\mathrm{d}n}{\mathrm{d}V} = -\begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & -f'(J) \\ \dfrac{-1}{nd} & 0 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} \dfrac{2K}{n^3} & 1 & 0 \\ 0 & 1 & -f'(J) \\ \dfrac{V}{n^2 d} & 0 & 1 \end{vmatrix} \tag{7}$$

Evaluation of the determinants, with minor substitutions from the foregoing definitions, yields a surprisingly simple result

$$\frac{\mathrm{d}n}{\mathrm{d}V} = \left\{ d \left[J - \frac{2C_P}{\left(\dfrac{\mathrm{d}C_P}{\mathrm{d}J}\right)} \right] \right\}^{-1} \tag{8}$$

Received Feb. 3, 1996; revision received May 10, 1996; accepted for publication May 13, 1996.

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Once again assumption 2 (i.e., $dC_P/dJ < 0$) guarantees that $dn/dV > 0$. Therefore, if V decreases (at constant throttle), then so must n. If on the contrary the pilot dives (again leaving the throttle alone), the engine speeds up. Equation (8), along with knowledge of $C_P(J)$ for values of J in the vicinity of the initial cruise condition, allows one to proceed stepwise to determine $n(V)$.

One Last Question

But why, one might reasonably ask, is the derivative of the power coefficient with respect to advance ratio always negative? In fact, if no restriction is placed on its domain of definition, it's not true that that derivative is always negative; for examples showing $dC_P/dJ > 0$, see the classic paper of Hartman and Biermann.[3] But it is true that within the fairly well-defined working range of advance ratio values, dC_P/dJ is indeed negative. Why?

The blade element theory of propeller action provides a hodgepodge of contributing conditions, but no direct and simple answer. Or at least this author hasn't found one. Employing the representative blade element analysis of von Mises,[4] however, a satisfactory solution is readily obtained. Von Mises argues that the power coefficient function is of the form

$$C_P = \lambda^2 \mu k \sin \beta'(J_2 - J) \tag{9}$$

where $\lambda \equiv \pi x$, μ is a solidity factor, β' is the blade setting angle (of the representative element at relative station x) with respect to the zero-lift direction, and J_2 is the value of advance ratio at which C_P goes negative. Since all of the numbers in Eq. (9) are positive, dC_P/dJ is negative.

Conclusions

Even in this advanced digital age, classical analytic reasoning can, on occasion, be parsimonious, instructive, and effective. But it seldom does the whole job.

References

1. Weick, F. E., *Aircraft Propeller Design*, McGraw–Hill, New York, 1930, p. 161.
2. Kaplan, W., *Advanced Calculus*, Addison–Wesley, Reading, MA, 1952, Chap. 2.
3. Hartman, E. P., and Biermann, D., "The Aerodynamic Characteristics of Full-Scale Propellers Having 2, 3, and 4 Blades of Clark Y and R.A.F. 6 Airfoil Sections," NACA TR-640, 1938, Fig. 18.
4. Von Mises, R., *Theory of Flight*, Dover, New York, 1959, p. 308.

Lifting-Line Theory of an Arched Wing in Asymmetric Flight

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Introduction

PRANDTL'S lifting-line theory[1,2] is, perhaps, the simplest among aerodynamic theories of high-AR wings in incompressible steady flows. At the essence of the theory, the wing is reduced to a line vortex of variable circulation located at its quarter-chord line; the wake behind the wing is reduced to a continua of noninteracting rectilinear trailing vortices, originating at the quarter-chord and extending to infinity. The equation governing the circulation of the quarter-chord vortex is obtained from the requirement that its local lift should be equal to that calculated from the effective angle of attack (AOA) of the respective wing section.

A recently developed asymptotic theory of high-AR arched wings[3] suggests that if the wing sweep is small enough, Prandtl's lifting-line theory can be extended so as to apply for arched (nonplanar) wings. Toward this end, the wing should be modeled by a planar arched vortex situated perpendicular to the direction in which the wake extends, regardless of the actual wing planform and its orientation relative to the flow. The (infinite) velocity induced by such a vortex on itself should be disregarded, both in constructing the equation governing the circulation of the wing vortex, and in computing the lift and drag forces acting on the wing.

This Note elucidates the extension of the classical lifting line theory for an arched wing of a typical gliding parachute.

Lifting-Line Formulation for an Arched Wing

Consider a parachute wing in a steady translatory motion through an (otherwise quiescent) incompressible fluid. Let u and ρ be the velocity of the wing and the density of the fluid, respectively. Consistent with the present design trends, the wing will be assumed to be unswept. It will be also assumed that in the front view the wing can be approximated by an arc of radius R and angle $2\phi_0$ (Fig. 1).

Let C be a right-handed Cartesian coordinate system, selected in such a way that its x axis coincides with the direction of the wing's motion, and the z axis points downward, perpendicular to the line connecting the left and right tips of the wing. Relative to C, the wing and its wake will be modeled by a curved line vortex in the (yz) plane, and by a cylindrical vortex sheet formed on the wing vortex and extending to infinity in the negative x direction. For the sake of being specific, it will be assumed henceforth that the x axis of C passes through the center of the wing's arc.

Let Γ, c, a, $\alpha_\perp$, and α_i, each defined on $(-\phi_0, \phi_0)$, be the sectional circulation, chord length, lift-slope coefficient, geometrical AOA, and induced AOA, respectively; $\alpha_\perp$ and α_i are shown in Fig. 1. By following the same arguments as those of the classical lifting-line theory, the circulation of the model wing is required to be such that its lift $\rho u \Gamma$ equals the lift $\frac{1}{2}\rho u^2 ca(\alpha_\perp - \alpha_i)$ of the actual wing; namely, for each ϕ in $(-\phi_0, \phi_0)$,

$$\Gamma(\phi) = \tfrac{1}{2} u c(\phi) a(\phi)[\alpha_\perp(\phi) - \alpha_i(\phi)] \tag{1}$$

Noting that the circulation of the wake vortices is $-[d\Gamma(\phi)/d\phi]\, d\phi$, one may use the Biot–Savart law to obtain[3,4]

$$\alpha_i(\phi) = \frac{1}{8\pi R u} \fint_{-\phi_0}^{\phi_0} \frac{d\Gamma(\phi')}{d\phi'} \cot \frac{\phi - \phi'}{2}\, d\phi' \tag{2}$$

Thus, for each ϕ in $(-\phi_0, \phi_0)$,

$$\frac{2\Gamma(\phi)}{uc(\phi)a(\phi)} + \frac{1}{8\pi R u} \fint_{-\phi_0}^{\phi_0} \frac{d\Gamma(\phi')}{d\phi'} \cot \frac{\phi - \phi'}{2}\, d\phi' = \alpha_\perp(\phi) \tag{3}$$

To obtain a unique solution for Γ, this equation should be supplemented by the edge conditions

$$\Gamma(-\phi_0) = \Gamma(\phi_0) = 0 \tag{4}$$

stating, on physical grounds, that no pressure differences may exist at the wingtips.

Received Jan. 21, 1996; revision received March 20, 1996; accepted for publication April 3, 1996.

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